# DYNAMIC BENDING OF POLYGONAL PLASTIC SLABS

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The study of the dynamic behavior of plastic structures under high-intensity short-term loads is important for evaluating the extent of their damage in order to determine the distances at which an explosion would not present a hazard. Such studies are also useful for determining the shape of products obtained from sheets formed by explosive forging. In such situations, the plastic strains are considerably greater in magnitude than the elastic strains. This makes it possible to take an ideal rigid-plastic body as an acceptable and convenient model. Studies of the dynamics of rigid-plastic structures currently embrace a wide range of problems and have been described in detail in [1-8]. As regards the bending of plates, many of the investigations have dealt with problems of the axisymmetric deformation of circular and annular plates and the construction of modifications [9-16] of the approximate solution obtained by Gvozdev for a rectangular slab in accordance with the "convert" scheme. This solution was obtained in [17] and was based on the assumption that the plastic-hinge lines are stationary. The modifications take several approaches. One approach is to use extreme principles and approximate integral estimates for the maximum residual deflection and time of motion with pressigned types of modal velocity fields [11, 14-16]. A second approach has been to allow for geometric nonlinearity [9, 14] within the framework of the scheme in [17]. A third direction that has been taken is to allow for the possibility of the development of a plastic zone while assuming that the plastic hinges are stationary [11-13]. As regards polygonal plates of general form, there are no solutions of the corresponding problems in the literature — even though the importance of obtaining such solutions was recognized as early as 1959 [9] and an appreciable number of approximate solutions has been found for static behavior [18].

Here, we use the approach taken by A. A. Gvozdev to construct a general solution for the dynamic bending of simply connected polygonal slabs. The individual sides of the contour are either hinged or fixed. The slabs are subjected to a shock load which is uniformly distributed over the surface. In the general case, it is assumed that the slab is at rest on an elastic, viscous, or viscoelastic base.

1. We will examine a plate made of an ideal rigid-plastic material with an arbitrary polygonal contour. The plate is loaded by a uniform distributed dynamic load of the intensity P(t). The plate is initially at rest on a viscoelastic base. The coefficients of elastic and viscous resistance of the plate material are $k_1^0$ and $k_2^0$. The individual sides of the contour may be hinged or fixed. In the case of loads which differ negligibly from the static limit loads, the scheme of motion of the plate will coincide with the scheme of static failure [17]. The static scheme takes the form of a set of rigid elements separated by linear plastic hinges. Given a sufficiently high level of loads — as in the case of bending of beams in [2, 4, 6] or circular annular plates in [5-8] — the dynamics of the polgyonal slab might be accompanied by the creation, growth, and disappearance of a zone of intensive plastic deformation. In connection with this, the general scheme of deformation of a polygonal plate can be represented as shown in Fig. 1, where $I_i$ (i = 1, 2,...,n) are the regions of the rigid planar elements and $I_p$ is the plastic zone. The equation of motion for the region $I_p$ has the form

$$\ddot{w} = p - k_1 w - k_2 \dot{w}, \quad p = Pt_0^2/\rho H, \quad w = W/H, \quad \tau = t/t_0, \quad k_1 = k_1^0 t_0^2/\rho, \quad k_2 = k_2^0 t_0/\rho, \tag{1.1}$$

where ρ is the surface density of the plate material; 2H is its thickness; $t_0$ is the characteristic time; the dot denotes differentiation with respect to the dimensionless time τ; W is the deflection.

We will examine the rotation of a rigid element $I_i$ ($A_iB_iC_iD_i$) around the bearing side $A_iD_i$ (Fig. 1). Proceeding on the basis of the condition of coincidence of the vertical velocities of points on the segment $B_iC_i$ on the side of the regions $I_i$ and $I_p$, it follows that $C_iB_i \parallel A_iD_i$, which in turn leads to

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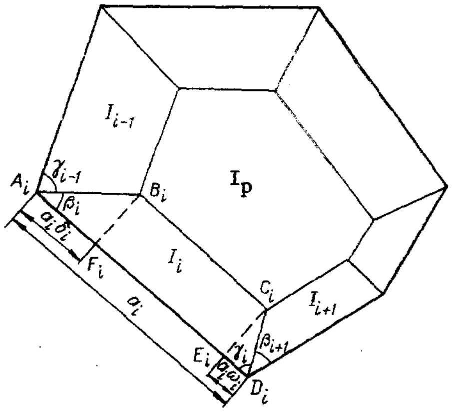

Fig. 1

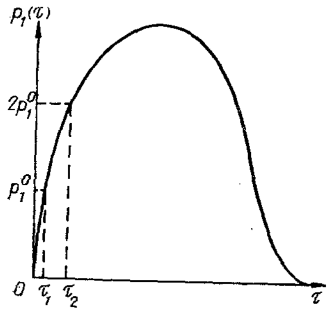

Fig. 2

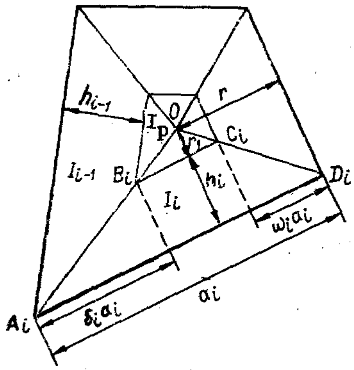

Fig. 3

$$\omega_i \operatorname{tg} \gamma_i = \delta_i \operatorname{tg} \beta_i; \tag{1.2}$$

$$\beta_i + \gamma_{i-1} = \varphi_i. \tag{1.3}$$

Here, $\varphi_i$ is the interior angle at the vertex $A_i$; $A_iD_i = a_i$; $B_iC_i = a_ib_i$; $A_iF_i = a_i\delta_i$; $E_iD_i = a_i\omega_i$. Also, it follows from the equality of the segment $C_iD_i$ for the elements $I_i$ and $I_{i+1}$ that

$$a_i\omega_i/\cos(\varphi_{i+1} - \beta_{i+1}) = a_{i+1}\delta_{i+1}/\cos\beta_{i+1}. \tag{1.4}$$

The equation of motion of the trapezoid $I_i$ around the reference line $A_iD_i$ will be written in the form

$$T_i\ddot{\alpha}_i = M_it_0^2, \tag{1.5}$$

where $T_i$ is the moment of inertia of the trapezoid $I_i$; $M_i$ is the total moment of the external forces applied to the surface and the contour of element $I_i$; $\alpha_i$ is the angle of deviation of $I_i$ from the horizontal. Considering that the segments $A_iB_i$, $B_iC_i$, $C_iD_i$ are linear plastic hinges with the limiting bending moment $M_0$ and taking into account that the bending moment on the line $A_iD_i$ is equal to zero when it is hinged and $-M_0$ when it is fixed, we represent Eq. (1.5) in the form

$$\delta_i^3 \operatorname{tg}^3 \beta_i [4 - 3(\delta_i + \omega_i)](\ddot{\alpha}_i + k_2\dot{\alpha}_i + k_1\alpha_i) =$$
$$= 2c_i[3 - 2(\delta_i + \omega_i)]\delta_i^2 \operatorname{tg}^2 \beta_i p(\tau) - d_i, \quad c_i = H/a_i, \quad d_i = 12M_0t_0^2(2 - \eta_i)/\rho a_i^3. \tag{1.6}$$

Here, $\eta_i = 0$ and 1 when the segment $A_iD_i$ is fixed or hinged, respectively. The condition of coincidence of the vertical velocities on the segment $B_iC_i$ determines the equality

$$\delta_i \operatorname{tg} \beta_i \dot{\alpha}_i = c_i\dot{w}. \tag{1.7}$$

The set of equations (1.1)-(1.4), (1.6), (1.7) constitutes the complete system of equations for finding the sought functions w, $\alpha_i$, $\delta_i$, $\omega_i$, $\beta_i$, $\gamma_i$ (i = 1, 2,...,n). The analysis and solution of the system for an arbitrary polygon is difficult and cumbersome. Thus, for simplicity, we will examine regular polygons with a contour having identically fixed sides in the absence of elastic and viscous resistance. In this case, $\varphi_i = \pi(n-2)/n = 2\varphi$, $\beta_i = \gamma_i = \varphi$, $a_i = 2a$, $\delta_i = \omega_i = \delta/2$, $\alpha_i = \alpha$, $\eta_i = \eta$ (i = 1, 2,...,n). Then the equations describing the dynamic behavior of a regular polygonal slab take the form

$$\delta^3(4 - 3\delta)\ddot{\alpha} = 2p_1\delta^2(3 - 2\delta) - m_0, \quad (\delta\dot{\alpha})^{\cdot} = p_1, \tag{1.8}$$

where $p_1 = pH/r$; $m_0 = 12M_0t_0^2(2 - \eta)/\rho r^3$; $r = a\tan\varphi$; r is the radius of a circle inscribed in the polygonal contour. The initial conditions for $\alpha$ are as follows: $\alpha(0) = \dot{\alpha}(0) = 0$.

To determine the static limit load, we need to put $\ddot{\alpha} = 0$ in the first equation of (1.8). Then we determine the kinematically possible limit load from the condition

$$p_1^0 = \min_{0<\delta\leqslant 1} p_1 = \min_{0<\delta\leqslant 1} m_0/2\delta^2(3 - 2\delta) = m_0/2. \tag{1.9}$$

Here, the plastic zone degenerates into the center of the inscribed circle, while the limit load in the case of hinged support coincides with the limit load for a circular plate of the radius r [8, 18]. The motion of a plate with a degenerate plastic region $I_p$ will be described by the first equation of system (1.8) with $\delta = 1$.

Let $p(\tau) \leqslant p_1^0$ ("low" loads) on the time interval $0 \leqslant \tau < \tau_1$ (first phase). Then the plate will remain in the undeformed state and at rest during this period of time.

We integrate the second equation of system (1.8) with allowance for the conditions

$$\dot{\alpha}(\tau_k) = \dot{\alpha}_k,\ \delta(\tau_k) = \delta_k:\ \delta\dot{\alpha} = I_k(\tau) + \delta_k\dot{\alpha}_k,\ I_k(\tau) = \int_{\tau_k}^{\tau} p_1(s)\,ds,\ k = 1, 2, \ldots$$

($\tau_k$ is a certain fixed moment of time in the interval of existence of the region $I_p$ in non-degenerate form). Using this equality and eliminating $\ddot{\alpha}$ from system (1.8), we obtain $[\delta^2(2-\delta)(I_k + \delta_k\dot{\alpha}_k)]^{\cdot} = m_0 = 2p_1^0$, from which

$$\delta^2(2-\delta) = [2p_1^0(\tau - \tau_k) + \delta_k^3(2-\delta_k)\dot{\alpha}_k](I_k + \delta_k\dot{\alpha}_k)^{-1}. \tag{1.10}$$

It follows from this equation that at k = 1

$$\lim_{\tau\to\tau_1} \delta^2(2-\delta) = \delta_1^2(2-\delta_1) = 2p_1^0/p_1(\tau_1), \tag{1.11}$$

from which it is evident that $\delta_1 < 1$ at $p_1(\tau_1) > 2p_1^0$ and $\delta_1 \geqslant 1$ at $p_1(\tau_1) \leqslant 2p_1^0$. Thus, if the load is such that $p_1(\tau_1) > 2p_1^0$ ("high" load), then the motion of the plate will begin in the presence of the zone $I_p$ and will be described by system (1.8) with the initial conditions $\delta = \delta_1$, $\alpha = \dot{\alpha} = 0$ [$\delta_1$ is determined from (1.11)].

Since by definition the value of $\delta$ cannot exceed unity, at a load $p_1^0 < p_1(\tau_1) \leqslant 2p_1^0$, it must be assumed that the motion of the plate will begin in the absence of a plastic zone and will be described by the first equation of (1.8) with $\delta = 1$.

Let us take a closer look at the motion of the plate under the influence of the load depicted in Fig. 2. In this case, with allowance for the initial conditions $\alpha(\tau_1) = \dot{\alpha}(\tau_1) = 0$, $\delta = 1$, in the second phase $(\tau_1 \leqslant \tau \leqslant \tau_2)$

$$\begin{gathered}\dot{\alpha}(\tau) = 2I_1(\tau) - 2p_1^0(\tau - \tau_1),\\ \alpha(\tau) = 2J_1(\tau) - p_1^0(\tau-\tau_1)^2,\quad J_k(\tau) = \int_{\tau_k}^{\tau} I_k(s)\,ds,\quad k = 1, 2, \ldots\end{gathered} \tag{1.12}$$

The end of this phase ($\tau = \tau_2$), corresponding to the beginning of formation of the plastic zone, will be determined as follows. It follows from (1.10) with k = 2, $\delta_2 = 1$ that

$$\dot{\delta}\delta(4-3\delta) = \{2p_1^0(I_2 + \dot{\alpha}_2) - [2p_1^0(\tau-\tau_2) + \dot{\alpha}_2]\,p_1(\tau)\}(I_2 + \dot{\alpha}_2)^{-2}. \tag{1.13}$$

It follows from (1.13) that at $p_1(\tau_2) = 2p_1^0$, $\dot{\delta}(\tau_2) = 0$, and $\dot{\delta}(\tau) < 0$, while at $p_1(\tau) > 2p_1^0$, $\delta(\tau) < 1$. Since $\delta \leqslant 1$, then $\delta$ begins to decrease. This corresponds to an increase in the size of the zone $I_p$ at the moment of time $\tau_2$, satisfying the condition $p_1(\tau_2) = 2p_1^0$.

The third phase of motion $(\tau_2 < \tau \leqslant \tau_3)$ occurs with a developed plastic zone and is described by system (1.8) with the initial conditions $\delta(\tau_2) = 1$, $\dot{\alpha}(\tau_2) = \dot{\alpha}_2$, $\alpha(\tau_2) = \alpha_2$. As a result of integration, we have [$\alpha_k = \alpha(\tau_k)$]

$$\begin{gathered}\dot{\alpha}(\tau) = (\dot{\alpha}_2 + I_2)\,\delta^{-1},\quad \alpha(\tau) = \alpha_2 + \int_{\tau_2}^{\tau}(\dot{\alpha}_2 + I_2)\,\delta^{-1}ds,\\ \delta^2(2-\delta) = [2p_1^0(\tau-\tau_2) + \dot{\alpha}_2](I_2 + \dot{\alpha}_2)^{-1}.\end{gathered} \tag{1.14}$$

It is evident from (1.13) that at $\tau > \tau_2$ and with satisfaction of the condition $F(\tau) < 0$ – where $F(\tau) = 2p_1^0[I_2(\tau) + \alpha_2] - [2p_1^0(\tau-\tau_2) + \dot{\alpha}_2]\,p_1(\tau)$ – the plastic zone increases in size. When $F(\tau) > 0$, it decreases, and at the moment $\tau_m$ such that $F(\tau_m) = 0$ it reaches its maximum size.

The time $\tau_3$ at which the phase ends is determined from the condition $\delta(\tau_3) = 1$ $(\tau_3 > \tau_2)$ and, in accordance with (1.14), satisfies the equation $I_2(\tau_3) = 2p_1^0(\tau_3 - \tau_2)$.

The fourth phase $(\tau_3 < \tau \leqslant \tau_4)$ is determined by the first equation of system (1.8) with $\delta = 1$. As a result, we obtain $\dot{\alpha}(\tau) = \dot{\alpha}_3 + 2I_3 - 2p_1^0(\tau - \tau_3)$, $\alpha(\tau) = \alpha_3 + 2J_3 - p_1^0(\tau-\tau_3)^2 + \dot{\alpha}_3(\tau - \tau_3)$. The end of the phase is found from the condition $\dot{\alpha}(\tau_4) = 0$ and is equal to $\tau_4 = \tau_1 + I_1(\tau_4)/p_1^0$.

The residual deflection at the center of the plate $W_f = r\int_{\tau_1}^{\tau_5}\delta\dot{\alpha}\,d\tau$.

In the case of "moderate" loads $p_1^0 < p_1(\tau) \leqslant 2p_1^0$, the time $\tau_2$ corresponds to the moment of stoppage of the plate, and during the period $\tau_1 \leqslant \tau \leqslant \tau_2$ the plate's motion is described by Eqs. (1.12). The time $\tau_2$ of stoppage is determined from $\dot{\alpha}(\tau_2) = 0$ and satisfies the equation $\tau_2 = \tau_1 + I_1(\tau_2)/p_1^0$.

For a high-intensity dynamic load which develops instantaneously and then decays, the first and second phases of motion in the above-described structure of the solution should be discarded and it is assumed that $\tau_1 = \tau_2 = 0$, $\dot{\alpha}(\tau_2) = 0$, $\alpha(\tau_2) = 0$. Here, the initial value $\delta_0 = \delta(0)$ is determined from the equality $\delta_0^2(2-\delta_0) = 2p_1^0/p_1(0)$, $p_1(0) \geqslant 2p_1^0$, obtained from (1.10) at $\tau_k = 0$ with the asymptote $\tau \to 0$. For a square pulse, $\delta(\tau) = \delta_0$ during the entire time of action of the pulse, and $\tau_m$ corresponds to the moment of unloading.

2. The solution described above is not difficult to modify in order to find the corresponding solutions for plates having a coaxial polygonal hole which is free or is reinforced by a rigid ring.

For plates with a free hole, characterized by the parameter $\delta$, the corresponding solutions can be obtained if we assume that the bending moments on the lines $B_iC_i$ are equal to zero. Then, with a fixed $\delta$, the equation of motion will be described by the first equation of (1.8) with the replacement of $m_0$ by $m_0' = 12M_0t_0^2(1-\eta+\delta)/\rho r^3$.

For plates with a rigid ring, characterized by the parameter $\delta$, linear plastic hinges and shearing forces will develop on the lines $B_iC_i$. These hinges and forces will influence the mutual motion of the rigid elements $I_i$ and $I_p$ (see Fig. 1). In the case of a regular polygonal contour, symmetry allows us to assume that the shearing forces are uniformly distributed over the contour of the internal hole. Taking this into account, we obtain the following equations of motion of the elements $I_p$ and $I_i$, respectively: $\ddot{w} + k_2\dot{w} + k_1w = [p + 2q/(1-q)]\rho_2$, $\delta^3(4-3\delta)(\ddot{\alpha} + k_2\dot{\alpha} + k_1\alpha)\tan\varphi = 2pc\delta^2(3-2\delta) + 12q\times c\delta(1-\delta) - m_0\tan\varphi$ ($c = H/a$, $q = Qt_0^2/\rho Ha\tan\varphi$, $\rho_2 = \rho/\rho_1$, Q is the shearing force, $\rho$ and $\rho_1$ are the densities of the materials of the plate and ring). Considering the equality of the vertical components of the velocities on the boundary of the plate and ring $c\dot{w} = \dot{\alpha}\delta\tan\varphi$ and eliminating q, we have an equation for the deflection w of the plate

$$(\ddot{w} + k_2\dot{w} + k_1w)\,[\rho_2\delta^2(4-3\delta) - 6\delta(1-\delta)^2] = 2p\rho_2\delta\,(9\delta - 5\delta^2 - 3) - m_0\rho_2\mathrm{tg}\,\varphi/c. \quad (2.1)$$

Since $\delta$ = const, Eq. (2.1) is easily integrated with the initial conditions $w(0) = \dot{w}(0) = 0$. Taking these conditions into account, we use (2.1) to find the limit load $p^0$, above which the plate will begin to move. In finding the limit load, we assume that $\ddot{w}(0) = 0$: $p^0 = m_0 \times \tan\varphi/[2\delta(9\delta - 5\delta^2 - 3)c]$.

3. One important class of plates consists of those plates which are identically fastened over their entire contour in the case where the contour is a convex polgyon described around a circle of the radius r. For such plates, the linear hinges in the limiting state intersect at the center of the circle [18]. Since $B_iC_i \parallel A_iD_i$ (Fig. 3, where $B_iC_i$ is a segment of the straight line separating the rigid element $I_i$ and the zone $I_p$), then we can inscribe a circle of radius $r_1$ in the contour of the region $I_p$. Since $h_i = r - r_1 = h_{i-1} = h$ = const, $\eta_i = \eta$, then from the similarity $\Delta A_iOD_i$ and $\Delta B_iOC_i$ we will have $r(\omega_i + \delta_i) = h$. Thus, $\delta_i + \omega_i = \delta = h/r$, and Eqs. (1.6) and (1.7) will take the form $\delta^3(4-3\delta)(\ddot{\alpha} + k_2\dot{\alpha} + k_1\alpha) = p_1\delta^2(3 - 2\delta) - m_0$, $r\delta\dot{\alpha} = H\dot{w}$. At $k_1 = k_2 = 0$, taking (1.1) into account, this system reduces to system (1.8). This means that the dynamic behavior of such plates will be similar to that of corresponding plates with a regular polygonal contour.

The solution proposed above can be used to solve the problem of optimizing the shape of a covering from the viewpoint of minimizing its damage, assuming constancy of the covered area, the thickness of the plate, and the method of fastening on all sides.

For a regular n-angle plate, the limit load $p_0^n = 6M_0(2-\eta)S^{-1}n\{\tan[\pi(n-2)/2n]\}^{-1}$ (S is the area of the plate). Since $p_0^n$ decreases with an increase in n, then $\min_{n\geqslant 3} p_0^n = \lim_{n\to\infty} p_0^n = 6M_0 \times (2-\eta)\pi/S$. It is evident from this that a circular plate of the radius $\sqrt{S/\pi}$ will have the minimum limit load, while a triangular plate will have the maximum limit load. Since the residual deflection and the time of motion of the plate are inversely proportional to the limit load, the circular plate will be subjected to the most damage. In the class of rectangular plates with a ratio of the sides $\gamma \geqslant 1$, the most damage is inflicted on square plates ($\gamma = 1$), since the limit load

$$p_0^\gamma = 6M_0(2-\eta)S^{-1}(1+\sqrt{1+3\gamma^2})^2(9\gamma)^{-1},$$

$$\text{and } \min_{\gamma\geqslant 1} p_0^\gamma = p_0^\gamma|_{\gamma=1} = 6M_0(2-\eta)S^{-1}.$$

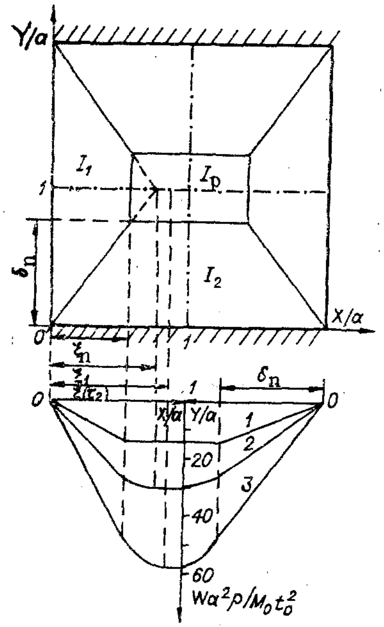


Fig. 4

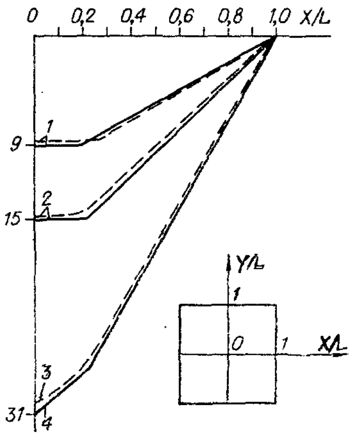


Fig. 5

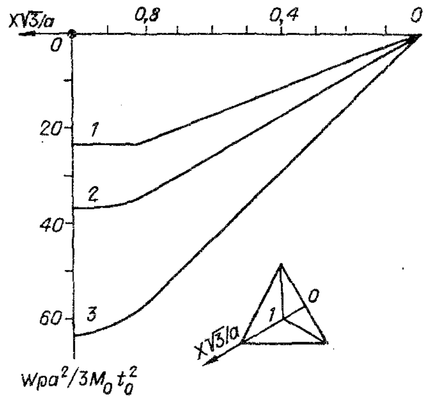


Fig. 6

4. As an example, we will examine the problem of the dynamic bending of a square plate. Two of the opposing sides of the plate are fixed, while the other two sides are either fixed ($\eta = 0$) or hinged ($\eta = 1$). The plate is subjected to a load in the form of a square pulse of intensity $p_1$ acting over the period of time $0 \leqslant \tau \leqslant 1$. The general scheme of deformation of such a plate is depicted in Fig. 4, where $\delta$ and $\xi$ are quantities characterizing the dimensions of the plastic region $I_p$, while $I_i$ represents the rigid zones ($i = 1, 2$). The equations of dynamic behavior have the form

$$\xi^3(4 - 3\delta)\ddot{\alpha}_1 = 2p_1\xi^2(3 - 2\delta) - (2 - \eta)m/2; \tag{4.1}$$

$$\delta^3(4 - 3\xi)\ddot{\alpha}_2 = 2p_1\delta^2(3 - 2\xi) - m; \tag{4.2}$$

$$(\xi\dot{\alpha}_1)^{\cdot} = p_1; \tag{4.3}$$

$$\xi\dot{\alpha}_1 = \delta\dot{\alpha}_2 \tag{4.4}$$

with $m = 24M_0/\rho a^3$. The limit load

$$p_1^0 = m(2 - \eta)/4\xi_0^2. \tag{4.5}$$

Here

$$\xi_0 = (-1 + \sqrt{1 + 6/(2 - \eta)})(2 - \eta)/2. \tag{4.6}$$

At $p_1 \leqslant p_1^0$ ("low" loads), no motion takes place. At $p_1^0 < p_1 \leqslant p_1^1$ ("moderate" loads), where

$$p_1^1 = m(2 - \eta)/(2\xi_1^2); \tag{4.7}$$

$$\xi_1 = (-1 + \sqrt{1 + 16/(2 - \eta)})(2 - \eta)/4, \tag{4.8}$$

motion will take place with $\delta = 1$ and $\xi$ determined from the equation $p_1 = m[4 - 3\xi - 2\xi^2/(2 - \eta)](2 - \eta)[4\xi^2(1 - \xi)]^{-1}$, with the degeneration of the zone $I_p$ into a segment of straight line. At $p_1 > p_1^1$ ("high" loads), the plate begins to move in the presence of a developed zone $I_p$.

Let us examine the "high" loads : $p_1 > p_1^1$. Here, in the first phase ($0 \leqslant \tau \leqslant 1$), with $p_1 = \mathrm{const}$, $\xi = \xi_n$, and $\delta = \delta_n$, we obtain the following from (4.1)-(4.4)

$$\dot{\alpha}_1(\tau) = p_1\tau/\xi_n,\ \alpha_1(\tau) = p_1\tau^2/2\xi_n,\ \dot{\alpha}_2(\tau) = p_1\tau/\delta_n,\ \alpha_2(\tau) = p_1\tau^2/2\delta_n,$$
$$\xi_n = 2[1 - m(2\delta_n^2 p_1)^{-1}],$$

where $\delta_n$ is determined from the equation

$$p_1 = m\left[\sqrt{16(2 - \delta_n)/(2 - \eta) + \delta_n^2} + \delta_n\right]^2(2 - \eta)[32(2 - \delta_n)\delta_n^2]^{-1}.$$

In the second phase ($1 < \tau \leqslant \tau_1$), the region $I_p$ undergoes compression, and, with $p_1 = 0$, $\delta = \delta(\tau)$, and $\xi = \xi(\tau)$, we obtain the following from (4.1)-(4.4)

$$\dot{\alpha}_1(\tau) = p_1/\xi(\tau), \quad \alpha_1(\tau) = \alpha_1(1) + p_1\int_1^\tau \xi^{-1}(s)\,ds, \quad \dot{\alpha}_2(\tau) = p_1/\delta(\tau),$$

$$\alpha_2(\tau) = \alpha_2(1) + p_1\int_1^\tau \delta^{-1}(s)\,ds,$$

where $\delta(\tau)$ and $\xi(\tau)$ are found from the equation $p_1 = m\tau[\sqrt{16(2-\delta)/(2-\eta)+\delta^2} + \delta]^2 \times (2-\eta)[32(2-\delta)\delta^2]^{-1}$, $\xi(\tau) = 2[1 - m\tau(2\delta^2 p_1)^{-1}]$. The moment of time $\tau = \tau_1$ is determined from the condition $\delta(\tau_1) = 1$ and is equal to $\tau_1 = p_1/p_1^1$. Here, $\xi(\tau_1) = \xi_1$ [$p_1^1$, $\xi_1$ are calculated from (4.7), (4.8)].

The third phase of motion ($\tau_1 < \tau \leqslant \tau_2$) occurs in the absence of the zone $I_p$. The solution is described by system (4.1), (4.2), (4.4) with $p_1 = 0$, $\delta = 1$ and the initial conditions $\xi(\tau_1) = \xi_1$, $\dot{\alpha}_i(\tau_1) = \dot{\alpha}_{i1}$, $\alpha_i(\tau_1) = \alpha_{i1}$ ($i = 1, 2$). System (4.1), (4.2), (4.4) is solved numerically by the Runge–Kutta method. The time at which the plate stops $\tau_2$ is found from the condition $\dot{\alpha}_i(\tau_2) = 0$ ($i = 1, 2$).

Calculations showed that, taking into account the rounding error, the relations $\dot{\xi}(\tau_2) = 0$, $I_1(\tau_2) = p_1^0\tau_2$, $2\xi_2^2 - (2-\eta)(3 - 2\xi_2) = 0$ are valid at the moment $\tau_2$. The residual deflection at the center of the plate $W_f = c\int_0^{\tau_2} \delta(\tau)\dot{\alpha}_2(\tau)\,d\tau$.

Figure 4 shows the distribution of the deflections of a plate with two hinged and two fixed edges subjected to the load $p_1^* = 1.5p_1^{1*}$ ($p_1^* = p_1\rho a^3/M_0t_0^2$, $p_1^{1*} = 19.69$) at the moments of time $\tau = 1$, $\tau_1 = 1.5$, $\tau_2 = 3.4$ (curves 1-3). Figure 5 quantitatively compares the results of calculations for a hinged plate (dashed lines) loaded by a uniformly distributed square pulse with the solution from [5] (solid lines) obtained by linear programming. It was assumed that $p_1^* = 3p_1^S$, where $p_1^* = p_1\rho L^3/M_0t_0^2$, 2L is the length of a side of the plate, and $p_1^S = 5.716$ is the limit load from [5]. It should be noted that for the given plate, the limit load determined in [19] by the simplex method is 5.784, while the limit load found from (1.9) is 6. Figure 5 shows the deflections of the plate at the moments of time $\tau = 1$ (unloading) and $\tau = 1.3$ and the residual deflections at the moments $\tau = 3.3$ [5] and $\tau = 2.87$ (above-derived formulas). It can be seen that the values are sufficiently close together, but the solution obtained here was considerably easier to find. It should also be noted that for a square pulse and a hinged square plate, the formulas obtained in the present study for residual deflection coincide with the formulas found in [10] on the basis of another, more complex solution.

As another example, let us examine the problem of a regular triangular hinged plate, with the side 2a, subjected to a uniformly distributed square pulse of the intensity $p_1$. In this case, the equations of motion have the form (1.8) $\delta^3(4 - 3\delta)\ddot{\alpha} = 2p_1\delta^2(3 - 2\delta) - m_0$, $(\delta\dot{\alpha})^\cdot = p_1$. The limit load $p_1^0 = m_0/2$. At $p_1^0 < p_1 \leqslant 2p_1^0$ ("moderate" loads), the motion of the plate takes place with three linear plastic hinges which coincide with the bisectors of the angles of the triangle. At $p_1 > 2p_1^0$ ("high" loads), motion occurs with the formation of a plastic zone in the form of a regular triangle similar to the contour triangle. Here, in the first phase of motion ($0 \leqslant \tau \leqslant 1$, $p_1 = \text{const}$, $\delta = \delta_n = \text{const}$) $\dot{\alpha}(\tau) = p_1\tau/\delta_n$, $\alpha(\tau) = p_1\tau^2/2\delta_n$, $\dot{w}(a\sqrt{3}/3, \tau) = p_1\tau$, $w(a\sqrt{3}/3, \tau) = p_1\tau^2/2$ [$w(x, \tau)$ is the deflection along the OX axis (Fig. 6)]. The size of the plastic zone $\delta_n$ is found from the equation

$$\delta_n^2(2 - \delta_n) = 2p_1^0/p_1.$$

In the second phase [$1 \leqslant \tau \leqslant \tau_1$, $p_1 = 0$, $\delta = \delta(\tau)$]

$$\dot{\alpha}(\tau) = p_1/\delta(\tau), \quad \alpha(\tau) = \alpha(1) + \int_1^\tau \delta^{-1}(s)\,ds, \quad w(a\sqrt{3}/3, \tau) = p_1(\tau - 1/2),$$

$$w(\delta_n a\sqrt{3}/3, \tau_1) = w(a\sqrt{3}/3, 1) + p_1\delta_n\int_1^{\tau_1} \delta^{-1}(s)\,ds = w(a\sqrt{3}/3, 1) +$$

$$+ p_1^2\delta_n(2p_1^0)^{-1}[4(1 - \delta_n) - 3(1 - \delta_n^2)/2],$$

where $\delta(\tau)$ is found from

$$\delta^2(2 - \delta) = 2p_1^0\tau/p_1; \quad \delta(\tau_1) = 1; \quad \tau_1 = p_1/2p_1^0.$$

In the third phase ($\tau_1 < \tau \leqslant \tau_2$, $p_1 = 0$, $\delta = 1$, $\tau_2$ being the time at which motion ends),

$$\dot{\alpha}(\tau) = 2(p_1 - p_1^0\tau), \quad \tau_2 = p_1/p_1^0, \quad \alpha(\tau) = \alpha(\tau_1) + 2\int_{\tau_1}^{\tau}(p_1 - p_1^0 s)\,ds,$$

$$w(a\sqrt{3}/3, \tau_2) = w(a\sqrt{3}/3, \tau_1) + 2\int_{\tau_1}^{\tau_2}(p_1 - p_1^0\tau)\,d\tau,$$

$$w(\delta_n a\sqrt{3}/3, \tau_2) = w(\delta_n a\sqrt{3}/3, \tau_1) + 2\delta_n\int_{\tau_1}^{\tau_2}(p_1 - p_1^0\tau)\,d\tau.$$

Figure 6 shows the distribution of the deflections at the moments of time $\tau = 1$, $\tau_1 = 1.3$, $\tau_2 = 2.6$ (curves 1-3) in section along the bisectors of the angle at $p_1^* = 2.6p_1^{*0}$ ($p_1^{*0} = 18$, $p_1^* = p_1\rho a^3 \tan^3\varphi/M_0t_0^2$).

## LITERATURE CITED


1. J. Reichert and J. Pearson, Explosive Forming of Metals [Russian translation], Mir, Moscow (1966).
2. I. L. Dikovich, Dynamics of Elastoplastic Beams [in Russian], Sudpromgiz, Leningrad (1962).
3. I. I. Gol'denblat and N. A. Nikolaenko, Design of Structures to Withstand Seismic Impulsive Forces [in Russian], Gosstroiizdat, Moscow (1960).
4. Kh. A. Rakhmatulin and Yu. A. Dem'yanov, Strength under Intensive Short-Term Loads [in Russian], Fizmatgiz, Moscow (1961).
5. M. I. Erkhov, Theory of Ideally Plastic Bodies and Structures [in Russian], Nauka, Moscow (1978).
6. K. L. Komarov and Yu. V. Nemirovskii, Dynamics of Rigid-Plastic Structural Elements [in Russian], Nauka, Novosibirsk (1984).
7. M. I. Reitman and G. S. Shapiro, "Dynamic theory of plasticity," in: Elasticity and Plasticity, VNIITI, Moscow (1968).
8. V. N. Mazalov and Yu. V. Nemirovskii, "Dynamics of thin-walled plates," in: Problems of the Dynamics of Elastoplastic Media. Mechanics. New Developments in Foreign Science [Russian translation], Mir, Moscow (1975).
9. N. A. Jones, T. O. Uran, and S. A. Tekin, "The dynamic plastic behavior of plates," Int. J. Solids Struct., 6, No. 12 (1970).
10. A. D. Cox and L. W. Morland, "Dynamic plastic deformation of simply supported square plates," J. Mech. Phys. Solids, 7, No. 4 (1959).
11. É. Birma, "Dynamics of plastic rectangular plates," Uch. Zap. Tartusk. Univ., No. 305 (1972).
12. J. Heinrich, "Intenzivni dynamicke zatiženi pravouhlych desek," Stavebnicky Čas., 13, No. 9 (1965).
13. L. Tsonev, Prilozhn. Mekh., 6, No. 1 (1971).
14. N. A. Jones, "A theoretical study of dynamic plastic behavior of beams and plates with finite deflections," Int. J. Solids Struct., 7, No. 8 (1971).
15. S. Kaliszky, "Approximate solutions for impulsive loaded plastic structures and continua," Int. J. Nonlinear Mech., 5, No. 1 (1970).
16. W. I. Morales and G. E. Nevill, "Lower bounds on deformation of dynamically loaded rigid-plastic continua," AIAA J., 8, No. 11 (1970).
17. A. A. Gvozdev, "Design of structures to withstand a blast wave," Stroit. Promst., Nos. 1-2 (1943).
18. A. R. Rzhanitsyn, Limit Equilibrium of Plates and Shells [in Russian], Nauka, Moscow (1983).
19. D. C. Koopman and R. H. Lance, "On linear programming and plastic limit analysis," J. Mech. Phys. Solids, 13, No. 2 (1965).